On the Quality of Wireless Network Connectivity

Soura Dasgupta  
Department of Electrical and Computer Engineering  
The University of Iowa

Guoqiang Mao  
School of Electrical and Information Engineering  
The University of Sydney  
National ICT Australia

Abstract—Despite intensive research in the area of network connectivity, there is an important category of problems that remain unsolved: how to measure the quality of connectivity of a wireless multi-hop network which has a realistic number of nodes, not necessarily large enough to warrant the use of asymptotic analysis, and has unreliable connections, reflecting the inherent unreliable characteristics of wireless communications? The quality of connectivity measures how easily and reliably a packet sent by a node can reach another node. It complements the use of capacity to measure the quality of a network in saturated traffic scenarios and provides a native measure of the quality of (end-to-end) network connections. In this paper, we explore the use of probabilistic connectivity matrix as a tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of connectivity are demonstrated. We show that the largest eigenvalue of the probabilistic connectivity matrix can serve as a good measure of the quality of network connectivity.

Index Terms—Connectivity, network quality, probabilistic connectivity matrix

I. INTRODUCTION

Connectivity is one of the most fundamental properties of wireless multi-hop networks [1]–[3], and is a prerequisite for providing many network functions. A network is said to be connected if and only if (iff) there is a (multi-hop) path between any pair of nodes. Further, a network is said to be k-connected iff there are k mutually independent paths between any pair of nodes that do not share any node in common except the starting and the ending nodes. k-connectivity is often required for robust operations of the network.

There are two general approaches to studying the connectivity problem. The first, spearheaded by the seminal work of Penrose [3] and Gupta and Kumar [1], is based on an asymptotic analysis of large-scale random networks, which considers a network of n nodes that are i.i.d. on an area with an underlying uniform distribution. A pair of nodes are directly connected iff their Euclidean distance is smaller than or equal to a given threshold r(n), independent of other connections. Some interesting results are obtained on the value of r(n) required for the above network to be asymptotically almost surely connected as n → ∞. In [4], [5], the authors extended the above results from the unit disk model to a random connection model, in which any pair of nodes separated by a displacement x are directly connected with probability g(x), independent of other connections. We refer readers to [7] for a more comprehensive review of related work.

The second approach is based on a deterministic setting and studies the connectivity and other topological properties of a network using algebraic graph theory. Specifically, consider a network with a set of n nodes. Its property can be studied using its underlying graph G(V, E), where V ≜ {v₁, . . . , vₙ} denotes the vertex set and E denotes the edge set. The underlying graph is obtained by representing each node in the network uniquely using a vertex and the converse. An undirected edge exists between two vertices iff there is a direct connection (or link) between the associated nodes

Define an adjacency matrix A₉ of the graph G(V, E) to be a symmetric n × n matrix whose (i, j)th, i ≠ j entry is equal to one if there is an edge between vᵢ and vⱼ and is equal to zero otherwise. Further, the diagonal entries of A₉ are all equal to zero. The eigenvalues of the graph G(V, E) are defined to be the eigenvalues of A₉. The network connectivity information, e.g., connectivity and k-connectivity, is entirely contained in its adjacency matrix. Many interesting connectivity and topological properties of the network can be obtained by investigating the eigenvalues of its underlying graph. For example, let µ₁ ≥ . . . ≥ µₙ be the eigenvalues of a graph G. If µ₁ = µ₂, then G is disconnected. If µ₁ = −µₙ and G is not empty, then at least one connected component of G is nonempty and bipartite. If the number of distinct eigenvalues of G is r, then G has a diameter of at most r − 1 [8]. Some researchers have also studied the properties of the underlying graph using its Laplacian matrix [9], where the Laplacian matrix of a graph G is defined as L₉ ≜ D − A₉ and D is a diagonal matrix with degrees of vertices in G on the diagonal. Particularly, the algebraic connectivity of a graph G is the second-smallest eigenvalue of L₉ and it is greater than 0 iff G is a connected graph. The algebraic connectivity quantifies the speed of convergence of consensus algorithms [10]. We refer readers to [8] for a comprehensive treatment of the topic.

Despite intensive research in the area, there is an important category of problems that remain unsolved: how to measure the quality of connectivity of a wireless multi-hop network.

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1In this paper, we limit our discussions to a simple graph (network) where there is at most one edge (link) between a pair of vertices (nodes) and an undirected graph.
network which has a realistic number of nodes, not necessarily large enough to warrant the use of asymptotic analysis, and has unreliable connections, reflecting the inherent unreliable characteristics of wireless communications. The quality of connectivity measures how easily and reliably a packet sent by a node can reach another node. It complements the use of capacity to measure the quality of a network in saturated traffic scenarios and provides a native measure of the quality of (end-to-end) network connections. In the following paragraphs, we elaborate on the above question using two examples.

Example 1: Consider a network with a fixed number of nodes with known transmission power to be deployed in a region. Assume that the wireless propagation model in that environment is known and its characteristics have been quantified through a priori measurements or empirical estimation. Further, a link exists between two nodes iff the received signal strength from one node at the other node is greater than or equal to a predetermined threshold and the same is also true in the opposite direction. One can then find the probability that a link exists between two nodes at two fixed locations: It is determined by the probability that the received signal strength is greater than or equal to the pre-determined threshold. Two related questions can be asked: a) If these nodes are deployed at a set of known locations, what is the quality of connectivity of the network, measured by the probability that there is a path between any two nodes, as compared to node deployment at another set of locations? b) How to optimize the node deployment to maximize the quality of connectivity?

Example 2: Consider a network with a fixed number of nodes. The transmission between a pair of nodes with a direct connection quantifying the inherent unreliable characteristics of wireless communications. There are no direct connections between some pairs of nodes because the probability of successful transmission between them is too low to be acceptable. How to measure the quality of connectivity of such a network, in the sense that a packet transmitted from one node can easily and reliably reach another node via a multi-hop path. Will a single “good” path between a pair of nodes be more preferable than multiple “bad” paths? These are further illustrated using Fig. 1 and 2.

In this paper, we explore the use of probabilistic connectivity matrix, a concept to be defined later in Section II, as a tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of connectivity are demonstrated. Based on the analysis, we show that the largest eigenvalue of the probabilistic connectivity matrix can serve as a good metric of the quality of network connectivity.

The rest of the paper is organized as follows. Section II defines the network settings, the probabilistic connectivity matrix and gives a method to compute the matrix. Section III introduces some inequalities associated with the entries of the probabilistic connectivity matrix. Section IV proves several important results about the probabilistic connectivity matrix. These directly associate the largest eigenvalue of the probabilistic connectivity matrix to the quality of connectivity and expose a structure that holds the promise of facilitating associated optimization tasks. Section V concludes the paper and discusses future work.

II. DEFINITION AND CONSTRUCTION OF THE PROBABILISTIC CONNECTIVITY MATRIX

Consider a network of \( n \) nodes. For some pair of nodes, an edge (or link) may exist with a non-negligible probability. The edges are undirected and independent.

Denote the underlying graph of the above network by \( G(V, E) \), where \( V = \{v_1, \ldots, v_n\} \) is the vertex set and \( E = \{e_1, \ldots, e_m\} \) is the edge set, which contains the set of
all possible edges. Here the vertices and the edges are indexed from 1 to \( n \) and from 1 to \( m \) respectively. For convenience, in some parts of this paper we also use the symbol \( e_{ij} \) to denote an edge between vertices \( v_i \) and \( v_j \) when there is no confusion. We associate with each edge \( e_i, \ i \in \{1, \ldots, m\} \), an indicator random variable \( I_i \) such that \( I_i = 1 \) if the edge \( e_i \) exists; \( I_i = 0 \) if the edge \( e_i \) does not exist. The indicator random variables \( I_{ij}, \ i \neq j \) and \( i, j \in \{1, \ldots, n\} \), are defined analogously.

In the following, we give a definition of the probabilistic adjacency matrix:

**Definition 1:** The probabilistic adjacency matrix of \( G(V,E) \), denoted by \( A_G \), is a \( n \times n \) matrix such that its \((i,j)\)th, \( i \neq j \), entry \( a_{ij} \overset{\Delta}{=} \Pr(I_{ij} = 1) \) and its diagonal entries are all equal to 1.

Due to the undirected property of an edge mentioned above, \( A_G \) is a symmetric matrix, i.e. \( a_{ij} = a_{ji} \). Note that the diagonal entries of \( A_G \) are defined to be 1, which is different from that common in the literature. This treatment of the diagonal entries can be associated with the fact that a node in the network can store a packet until better transmission opportunity arises when it finds the wireless channel busy [11].

The probabilistic connectivity matrix is defined in the following:

**Definition 2:** The probabilistic connectivity matrix of \( G(V,E) \), denoted by \( Q_G \), is a \( n \times n \) matrix such that its \((i,j)\)th, \( i \neq j \), entry is the probability that there exists a path between vertices \( v_i \) and \( v_j \), and its diagonal entries are all equal to 1.

As a ready consequence of the symmetry of \( A_G \), \( Q_G \) is also a symmetric matrix.

Given the probabilistic adjacency matrix \( A_G \), the probabilistic connectivity matrix \( Q_G \) is fully determined. However the computation of \( Q_G \) is not trivial because for a pair of vertices \( v_i \) and \( v_j \) there may be multiple paths between them and some of them may share common edges, i.e. are not independent. In the following paragraph, we give an approach to computing the probabilistic connectivity matrix.

Let \( (I_1, \ldots, I_m) \) be a particular instance of the indicator random variables associated with an instance of the random edge set. Let \( Q_G|\{I_1, \ldots, I_m\} \) be the connectivity matrix of \( G \) conditioned on \((I_1, \ldots, I_m)\). The \((i,j)\)th entry of \( Q_G|\{I_1, \ldots, I_m\} \) is either 0, when there is no path between \( v_i \) and \( v_j \), or 1 when there exists a path between \( v_i \) and \( v_j \). The diagonal entries of \( Q_G|\{I_1, \ldots, I_m\} \) are always 1. Conditioned on \((I_1, \ldots, I_m)\), \( G(V,E) \) is just a deterministic graph. Therefore the entries of \( Q_G|\{I_1, \ldots, I_m\} \) can be efficiently computed using a search algorithm, such as breadth-first search. Given \( Q_G|\{I_1, \ldots, I_m\} \), \( Q_G \) can be computed using the following equation:

\[
Q_G = E(Q_G|\{I_1, \ldots, I_m\})
\]  
(1)

where the expectation is taken over all possible instances of \((I_1, \ldots, I_m)\).

The approach suggested in the last paragraph is essentially a brute-force approach to computing \( Q_G \). A more efficient algorithm is suggested in Section IV.

**Remark 1:** A major difference between the (probabilistic) connectivity matrix and the adjacency matrix (or the Laplacian matrix) is that the latter matrix focuses on quantifying the relation between node pairs directly connected by an edge only while the former matrix focuses on quantifying the end-to-end relationship between node pairs. It is not trivial to obtain the connectivity matrix from the adjacency matrix or use the adjacency matrix to study network properties easily obtainable using the connectivity matrix.

**Remark 2:** For simplicity, the terms used in our discussion are based on the problems in Example 1. The discussion however can be easily adapted to the analysis of the problems in Example 2. For example, if \( a_{ij} \) is defined to be the probability that a transmission between nodes \( v_i \) and \( v_j \) is successful, the \((i,j)\)th entry of the probabilistic connectivity matrix \( Q_G \) computed using (1) then gives the probability that a transmission from \( v_i \) to \( v_j \) via a multi-hop path is successful under the best routing algorithm, which can always find a shortest and error-free path from \( v_i \) to \( v_j \) if it exists, or alternatively, the probability that a packet broadcast from \( v_i \) can reach \( v_j \) where each node receiving the packet only broadcasts the packet once. Therefore the \((i,j)\)th entry of \( Q_G \) can be used as a quality measure of the end-to-end paths between \( v_i \) and \( v_j \), which takes into account the fact that availability of extra paths between a pair of nodes can be exploited to improve the probability of successful transmissions.

**III. Some Key Inequalities for Connection Probabilities**

The entries of the probabilistic connectivity matrix give a measure of the quality of end-to-end paths. In this section, we provide some important inequalities that may facilitate further analysis of the quality of connectivity. Some of these inequalities are exploited in the next section to establish several key properties of the probabilistic connection matrix itself. We first introduce some results that are required for the further analysis of the probabilistic connectivity matrix \( Q_G \).

For a random graph with a given set of vertices, a particular event is *increasing* if the event is preserved when more edges are added into the graph. An event is *decreasing* if its complement is increasing.

Denote by \( \xi_{ij} \) the event that there is a path between vertices \( v_i \) and \( v_j \), \( i \neq j \). Denote by \( \xi_{ikj} \) the event that there is a path between vertices \( v_i \) and \( v_j \) and that path passes through the third vertex \( v_k \), where \( k \in \Gamma_n \setminus \{i,j\} \) and \( \Gamma_n \) is the set of indices of all vertices. Denote by \( \eta_{ij} \) the event that there is an edge between vertices \( v_i \) and \( v_j \). Denote by \( \Pi_{ikj} \) the event that there is a path between vertices \( v_i \) and \( v_k \) and there is a path between vertices \( v_k \) and \( v_j \), where \( k \in \Gamma_n \setminus \{i,j\} \). It can be shown from the above definitions that

\[
\xi_{ij} = \eta_{ij} \cup (\cup_{k \neq i,j} \xi_{ikj})
\]
(2)

Let \( q_{ij}, \ i \neq j \), be the \((i,j)\)th entry of \( Q_G \), i.e. \( q_{ij} = \Pr(\xi_{ij}) \). The following lemma can be readily obtained from the FKG inequality [6, Theorem 1.4] and the above definitions.
Lemma 1: For two distinct indices $i, j \in \Gamma_n$ and $\forall k \in \Gamma_n \setminus \{i, j\}$

$$q_{ij} \geq \max_{k \in \Gamma_n \setminus \{i, j\}} q_{ik} q_{kj}$$  \hspace{1cm} (3)

Proof: It follows readily from the above definitions that the event $\xi_{ij}$ is an increasing event. Using the FKG inequality:

$$\Pr (\xi_{ij}) \geq \Pr (\pi_{ikj}) = \Pr (\xi_{ik} \cap \xi_{kj}) \geq \Pr (\xi_{ik}) \Pr (\xi_{kj})$$  \hspace{1cm} (4)

Lemma 1 gives a lower bound of $q_{ij}$. The following lemma gives an upper bound of $q_{ij}$:

Lemma 2: For two distinct indices $i, j \in \Gamma_n$ and $\forall k \in \Gamma_n \setminus \{i, j\}$,

$$q_{ij} \leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \setminus \{i, j\}} (1 - q_{ik} q_{kj})$$  \hspace{1cm} (5)

where $a_{ij} = \Pr (\eta_{ij})$.

Proof: We will first show that $\xi_{ikj} \Leftrightarrow \xi_{ik} \cap \xi_{kj}$. That is, the occurrence of the event $\xi_{ikj}$ is a sufficient and necessary condition for the occurrence of the event $\xi_{ik} \cap \xi_{kj}$, where for two events $A$ and $B$, $A \cap B$ denotes the event that there exist two disjoint sets of edges such that the first set of edges guarantees the occurrence of $A$ and the second set of edges guarantees the occurrence of $B$.

Using the definition of $\xi_{ikj}$, occurrence of $\xi_{ikj}$ means that there is a path between vertices $v_i$ and $v_j$ and that path passes through vertex $v_k$. It follows that there exist a path between vertex $i$ and vertex $v_k$ and a path between vertex $v_k$ and vertex $v_j$ and the two paths do not have edge(s) in common. Otherwise, it will contradict the definition of $\xi_{ikj}$, particularly as the definition of a path requires the edges to be distinct. Therefore $\xi_{ikj} \Rightarrow \xi_{ik} \cap \xi_{kj}$. Likewise, $\xi_{ikj} \Leftrightarrow \xi_{ik} \cap \xi_{kj}$ also follows directly from the definitions of $\xi_{ikj}$, $\xi_{ik}$, $\xi_{kj}$ and $\xi_{ik} \cap \xi_{kj}$. Consequently

$$\Pr (\xi_{ikj}) = \Pr (\xi_{ik} \cap \xi_{kj}) \leq \Pr (\xi_{ik}) \Pr (\xi_{kj})$$  \hspace{1cm} (6)

where the inequality is a direct result of the BK inequality [6].

With a little bit abuse of the terminology, in the following derivations we also use $\xi_{ikj}$ to represent the set of edges that make the event $\xi_{ikj}$ happen, and use $\eta_{ij}$ to denote the edge between vertices $v_i$ and $v_j$.

Note that the set of edges $\bigcup_{k \in \Gamma_n \setminus \{i, j\}} \xi_{ikj}$ does not contain $\eta_{ij}$. Therefore using (2) and independence of edges (used in the third step)

$$q_{ij} = \Pr \left( \eta_{ij} \cup \left( \bigcup_{k \in \Gamma_n \setminus \{i, j\}} \xi_{ikj} \right) \right)$$

$$= 1 - \Pr \left( \eta_{ij} \cap \left( \bigcup_{k \in \Gamma_n \setminus \{i, j\}} \xi_{ikj} \right) \right)$$

$$= 1 - (1 - a_{ij}) \Pr \left( \bigcap_{k \in \Gamma_n \setminus \{i, j\}} \xi_{ikj} \right)$$

$$\leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \setminus \{i, j\}} \Pr (\xi_{ikj})$$

$$= 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \setminus \{i, j\}} (1 - \Pr (\xi_{ikj}))$$

$$\leq 1 - (1 - a_{ij}) \prod_{k \in \Gamma_n \setminus \{i, j\}} (1 - q_{ik} q_{kj})$$  \hspace{1cm} (7)

where in (7), FKG inequality and the fact that $\overline{\xi_{ikj}}$ is a decreasing event are used and the last step results due to (6).

When there is no edge between vertices $v_i$ and $v_j$, which is the generic case, the upper and lower bounds in Lemmas 1 and 2 reduce to

$$\max_{k \in \Gamma_n \setminus \{i, j\}} q_{ik} q_{kj} \leq q_{ij} \leq 1 - \prod_{k \in \Gamma_n \setminus \{i, j\}} \left(1 - q_{ik} q_{kj}\right)$$  \hspace{1cm} (9)

The above inequality sheds insight on how the quality of paths between a pair of vertices is related to the quality of paths between other pairs of vertices. It can be possibly used to determine the most effective way of improving the quality of a particular set of paths by improving the quality of a particular (set of) edge(s), or equivalently what can be reasonably expected from an improvement of a particular edge on the quality of end-to-end paths.

The following lemma further shows that relation among entries of the path matrix $Q_{ij}$ can be further used to derive some topological information of the graph.

Lemma 3: If $q_{ij} = q_{ik} q_{kj}$ for three distinct vertices $v_i, v_j$ and $v_k$, the vertex set $V$ of the underlying graph $G(V, E)$ can be divided into three non-empty and non-intersecting sub-sets $V_1, V_2$ and $V_3$ such that $v_i \in V_1$, $v_j \in V_3$ and $V_2 = \{v_k\}$ and any possible path between a vertex in $V_1$ and a vertex in $V_2$ must pass through $v_k$. Further, for any pair of vertices $v_l$ and $v_m$, where $v_l \in V_1$ and $v_m \in V_3$, $q_{lm} = q_{ik} q_{km}$.

Proof: Using (4) in the second step, it follows that

$$q_{ij} = \Pr \left( \left( \xi_{ij} \setminus \pi_{ikj} \right) \cup \pi_{ikj} \right) = \Pr (\xi_{ij} \setminus \pi_{ikj}) + \Pr (\pi_{ikj})$$

$$\geq \Pr (\xi_{ij} \setminus \pi_{ikj}) + q_{ik} q_{kj}$$

Therefore $q_{ij} = q_{ik} q_{kj}$ implies that $\Pr (\xi_{ij} \setminus \pi_{ikj}) = 0$ or equivalently $\xi_{ij} \Leftrightarrow \pi_{ikj}$.

Further, $\Pr (\xi_{ij} \setminus \pi_{ikj}) = 0$ implies that a possible path (i.e. a path with a non-zero probability) connecting $v_i$ and $v_k$ and a possible path connecting $v_k$ and $v_j$ cannot have any edge in common. Otherwise a path from $v_i$ to $v_j$, bypassing $v_k$, exists with a non-zero probability which implies $\Pr (\xi_{ij} \setminus \pi_{ikj}) > 0$. The conclusion follows readily that if $q_{ij} = q_{ik} q_{kj}$ for three distinct vertices $v_i, v_j$ and $v_k$, the vertex set $V$ of the underlying graph $G(V, E)$ can be divided into three non-empty and non-overlapping sub-sets $V_1, V_2$ and $V_3$ such that $v_i \in V_1$, $v_j \in V_3$ and $V_2 = \{v_k\}$ and a path between a vertex in $V_1$ and a vertex in $V_2$, if exists, must pass through $v_k$.

Further, for any pair of vertices $v_l$ and $v_m$, where $v_l \in V_1$ and $v_m \in V_3$, it is easily shown that $\Pr (\xi_{lm} \setminus \pi_{ikm}) = 0$. Due to independence of edges and further using the fact that $\Pr (\xi_{lm} \setminus \pi_{ikm}) = 0$, it can be shown that

$$\Pr (\xi_{lm}) = \Pr (\pi_{ikm}) = \Pr (\xi_{ik} \cap \pi_{km}) = \Pr (\xi_{ik}) \Pr (\pi_{km})$$

where the last step results because under the condition of $\Pr (\xi_{lm} \setminus \pi_{ikm}) = 0$, a path between $v_l$ and $v_k$ and a path between $v_k$ and $v_m$ cannot possibly have any edge in common.

An implication of Lemma 3 is that for any three distinct vertices, $v_i, v_j$ and $v_k$, if a relationship $q_{ij} = q_{ik} q_{kj}$ holds, vertex $v_k$ must be a critical vertex whose removal will render the graph disconnected.

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IV. PROPERTIES OF THE CONNECTIVITY MATRIX

Having established some inequalities obeyed by the entries of \( Q_G \), we now turn to establishing a measure of the quality of network connectivity. At the core of the development in this section is the following result.

Lemma 4: Each off-diagonal entry of the probabilistic connectivity matrix \( Q_G \) is a multiaffine function of \( a_{ij}, i \in \{1, \ldots, n\}, j > i \).

Proof: Observe that \( a_{ij} = \Pr(\eta_{ij}) \) and the events \( \eta_{ij}, i \in \{1, \ldots, n\}, j > i \) are independent. The conclusion in the lemma follows readily from the fact that the event associated with each \( q_{ij} \), i.e. there exists a path between vertices \( v_i \) and \( v_j \), is a union of intersections of these events \( \eta_{ij}, i \in \{1, \ldots, n\}, j > i \).

Due to the above multiaffine property, for any four positive integers \( k, l, i, j \in \{1, \ldots, n\} \), where \( p \neq q \) and \( i \neq j \), the following holds:

\[
q_{lk} = f(E \setminus \{e_{ij}\}) a_{ij} + g(E \setminus \{e_{ij}\})
\]

where \( f(E \setminus \{e_{ij}\}) \) and \( g(E \setminus \{e_{ij}\}) \) are non-negative constants within \([0, 1]\) determined by the state of the set of edges excluding \( e_{ij} \). \( g(E \setminus \{e_{ij}\}) = 0 \) implies that non-existence of the edge \( e_{ij} \) will render the vertices \( v_l \) and \( v_k \) disconnected. \( f(E \setminus \{e_{ij}\}) = 0 \) implies that the state of the edge \( e_{ij} \) is irrelevant for the end-to-end paths between \( v_l \) and \( v_k \). Further, \( f(E \setminus \{e_{ij}\}) \) can be used to measure the criticality of the edge \( e_{ij} \) to the end-to-end paths between \( v_l \) and \( v_k \).

Remark 3: Using the multiaffine property, a more efficient algorithm for computing \( Q_G \) than the one suggested earlier using (1) can be constructed. Particularly, the probabilistic connectivity matrix of a network forming a tree can be easily computed. Therefore the algorithm may start by first identifying a spanning tree in \( G(V, E) \) and computing the associated probabilistic connectivity matrix. Then, the edges in \( E \) out the spanning tree can be added recursively and the corresponding probabilistic connectivity matrix updated using (10).

We comment later in Remark 5 on how the multiaffine structure is also potentially useful for performing some of the optimization tasks inherent in maximizing connectivity. e.g. determination of the link whose improvement will bring the maximum benefit on connectivity.

A very desirable property of \( Q_G \) is established below.

Theorem 1: The probabilistic connectivity matrix \( Q_G \) is a positive semi-definite matrix. Further, \( Q_G \) is positive semi-definite but not positive definite iff there exist distinct \( i, j \in \{1, \ldots, n\} \), such that \( q_{ij} = 1 \).

The proof is omitted due to space limitation.

Let \( \lambda_1 \geq \cdots \geq \lambda_n \) be the eigenvalues of \( Q_G \). Note that \( \lambda_1 + \cdots + \lambda_n = n \). As an easy consequence of Theorem 1, \( n \geq \lambda_1 \geq 1 \) and \( 1 \geq \lambda_n \geq 0 \). In the best case, \( Q_G \) is a matrix with all entries equal to 1. Then \( \lambda_1 = n \) and \( \lambda_2 = \cdots = \lambda_n = 0 \). In the worst case, \( Q_G \) is an identity matrix. Then \( \lambda_1 = \cdots = \lambda_n = 1 \). This suggests that \( \lambda_1 \), i.e. the largest eigenvalue of \( Q_G \), can be used as a measure of quality of network connectivity and a larger \( \lambda_1 \) indicates a better quality.

Further, let \( X \) be a vector representing the number of packets broadcast by each node to the rest of the network and let \( Y \) be a vector representing the random number of packets received by each node. It is obvious that \( E[Y|X] = QGX \) then represents the expected number of packets received by each node. Using the property that \( Q_G \) is a symmetric matrix, it can be shown that

\[
\max_{\|X\|_2=1} \|E[Y|X]\|_2 = \max_{\|X\|_2=1} \|QGX\|_2 = \max_{\|X\|_2=1} \sqrt{X^TQ_G^TQ_GX} = \max_{\|X\|_2=1} \sqrt{X^TQ_G^2X} = \sqrt{\lambda_{\max}(Q_G^2)} = \lambda_{\max}(Q_G)
\]

where \( \lambda_{\max}(Q_G) \) is the maximum eigenvalue of \( Q_G \) and \( \|X\|_2 \) denotes the \( L^2 \)-norm or Euclidean norm of \( X \).

We will make this idea that \( \lambda_{\max}(Q_G) \) serves as a good measure of the quality of network connectivity more concrete in the following analysis. We start our discussion with a connected network and then extend to more generic cases. We will call a network connected if for all \( i, j \in \{1, \ldots, n\} \), \( q_{ij} > 0 \). Obviously the probabilistic connectivity matrix of a connected network is irreducible [12, p. 374] as all the entries of the matrix are non-zero. As a measure of the quality of network connectivity, if the path probabilities \( q_{ij} \) increase, the largest eigenvalue of the probabilistic connectivity matrix should also increase. This is formally stated below:

Theorem 2: Let \( G(V, E) \) and \( G'(V, E') \) be the underlying graphs of two connected networks defined on the same vertex set \( V \) but with different link probabilities. Let \( Q_G \) and \( Q_{G'} \) be the probabilistic connectivity matrices of \( G \) and \( G' \) respectively. If \( Q_G - Q_{G'} \) is a non-zero, non-negative matrix, then \( \lambda_{\max}(Q_G) < \lambda_{\max}(Q_{G'}) \).

Proof:

We need the following lemma to prove Theorem 2.

Lemma 5: Suppose \( A = A^T \neq B = B^T \) are non-negative, irreducible, real matrices, and \( B - A \) is a non-zero, non-negative matrix. Then: \( \lambda_{\max}(A) < \lambda_{\max}(B) \).

Proof: Observe at least one element of \( B - A \) is positive. From Perron-Frobenius theorem [12, p. 536], \( x \in \mathbb{R}^n \), the eigenvector corresponding to the largest eigenvalue of \( A \) can be chosen to have all elements positive. Then the result follows from the fact that:

\[
\lambda_{\max}(A)x^Tx = x^TAx = x^TBx - x^T(B-A)x < x^TBx \leq \lambda_{\max}(B)x^Tx
\]

as \( B - A \) is a non-zero, non-negative matrix.

Turning to the proof of Theorem 2 we note that the result follows directly from Lemma 5 and the fact \( Q_{G'} \) and \( Q_G \) satisfy the requirements of \( B \) and \( A \), respectively.

If the network is not connected, i.e. some entries of its probabilistic connectivity matrix is 0, the network can be

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2 A multiaffine function is affine in each variable when the other variables are fixed.

3 A matrix is non-negative if all its entries are greater than or equal to 0.
decomposed into disjoint components. Let the total number of components in the network be k. Let \( G_i \) be the subgraph induced on the set of vertices in the \( i^{th} \) component and \( Q_{G_i} \) be the probabilistic connectivity matrix of \( G_i \). It follows that

\[
\lambda_{\text{max}}(Q_G) = \max\{\lambda_{\text{max}}(Q_{G_1}), \ldots, \lambda_{\text{max}}(Q_{G_k})\}
\]  

(11)

We consider two basic situations: a) there are increases in some entries of \( Q_G \) from non-zero values but such increases do not change the number of components in the network. It then follows easily from Theorem 2 that \( \lambda_{\text{max}}(Q_{G_i}) > \lambda_{\text{max}}(Q_G) \). Depending on whether \( \lambda_{\text{max}}(Q_{G_i}) \) is greater than \( \lambda_{\text{max}}(Q_G) \) or not however, \( \lambda_{\text{max}}(Q_G) \) may or may not increase. b) there are increases in some entries of \( Q_G \) from zero to non-zero values and such increases reduce the number of components in the network. For situation b), we consider a simplified scenario where increases in the path probabilities merge two originally disjoint components, denoted by \( G_i \) and \( G_j \). The more complicated scenario where increases in the path probabilities join more than two originally disjoint components can be obtained recursively as an extension of the above simplified scenario. Let \( G' \) be the underlying graph of the network after increases in path probabilities and let \( G'_{ij} \) be the subgraph in \( G' \) induced on the vertex set \( V_i \cup V_j \). Obviously \( Q_{G'_{ij}} \) is an irreducible matrix and the following result can be established.

Lemma 6: Under the above settings,

\[
\lambda_{\text{max}}(Q_{G'_{ij}}) > \lambda_{\text{max}}(\text{diag}\{Q_{G_i}, Q_{G_j}\})
\]

(12)

The proof of Lemma 6 is straightforward and hence omitted.

Thus indeed the largest eigenvalues of the probabilistic connection matrices associated with disjoint components measure the quality of the components connection.

Remark 4: To compare two networks with different number of nodes, the normalized maximum eigenvalue of the probabilistic connectivity matrix, where the maximum eigenvalue is divided by the number of nodes, can be used.

Remark 5: The fact that the largest eigenvalue of the probabilistic connectivity matrix measures connectivity, suggests the following obvious optimization. Modify one or more \( a_{ij} \) under suitable constraints to maximize the largest eigenvalue of the probabilistic connectivity matrix. Results in [13] and [14] suggest that the multifunction dependence of the \( q_{ij} \) on the \( a_{ij} \), together with the fact that \( Q_G \) is positive semi-definite promise to facilitate such optimization.

V. CONCLUSIONS AND FURTHER WORK

In this paper we explored the use of the probabilistic connectivity matrix as a tool to measure the quality of network connectivity. Some interesting properties of the probabilistic connectivity matrix and their connections to the quality of network connectivity were demonstrated. Particularly, the off-diagonal entries of the probabilistic connectivity matrix provide a measure of the quality of end-to-end connections and we have also provided theoretical analysis supporting the use of the largest eigenvalue of the probabilistic connectivity matrix as a measure of the quality of overall network connectivity. Inequalities between the entries of the probabilistic connectivity matrix were established. These may provide insights into the correlations between quality of end-to-end connections. Further, the probabilistic connectivity matrix was shown to be a positive semi-definite matrix and its off-diagonal entries are multifunctions of link probabilities. These two properties are expected to be very helpful in optimization and robust network design, e.g., determining the link whose quality improvement will result in the maximum gain in network quality, and determining quantitatively the relative criticality of a link to either a particular end-to-end connection or to the entire network.

The results in the paper rely on two main assumptions: the links are symmetric and independent. We expect that our analysis can be readily extended such that the first assumption on symmetric links can be removed – in fact the results in Section III do not need this assumption. While in the asymmetric case the probabilistic connectivity matrix is no longer guaranteed to be positive semi-definite, we conjecture that the largest eigenvalue retains its significance. Discarding the second assumption requires more work. However, we are encouraged by the following observation. If we introduce conditional edge probabilities into the mix, then \( Q_G \) is still a multifunction of the \( a_{ij} \) and the conditional probabilities. Thus we still expect all the results in Section IV to hold, though the proof may be non-trivial. In real applications link correlations may arise due to both physical layer correlations and correlations caused by traffic congestion.

REFERENCES